Lecture 1

Three units in the course:

- Flag varieties of $\mathbb{R}^n$, $\mathbb{C}^n$, and general semisimple Lie alg / grps.
- Topology and complex geometry of flag varieties
  (Schubert cells etc.)
- Dynamics on flag varieties
  (elements of G and discrete subgrps of G)

See the course web page for books. None req'd.

Registered students: 20 min presentation at the end
Topic of choice (no precise dupes)
Final exam week in designated period.

Projective spaces and Grassmannians.

V a vector space over a field k, finite dimension.

$\mathbb{P}(V) = \{\text{1-dim subspaces of } V\}$ (also $\mathbb{P}_k(V)$ or $k\mathbb{P}(V)$)

$\mathbb{P}^*(V) = \{\text{codim-1 subsp of } V\}$ (Always mean k-vec subspace)

Common generalization: $Gr(d, V) = \{d\text{-dimensional subsp of } V\}$

Grassmannian of d-planes in V

So if $\dim_k V = n$, $Gr(1, V) = \mathbb{P}(V)$ and
$Gr(n-1, V) = \mathbb{P}^*(V)$.

Now let $k = \mathbb{R}$ or $\mathbb{C}$, so every k-vec space has a natural smooth mfld structure.

e.g. If V, W are k-vec spaces, then

$$\underbrace{\operatorname{Hom}(V,W)}_{k\text{-lin maps}} \underset{k\text{-vec iso}}{\cong} k^{(\dim V)(\dim W)} \underset{\text{diffeo}}{\cong} \mathbb{R}^N \qquad N = \begin{cases} \dim V \dim W & k = \mathbb{R} \\ 2 \dim V \dim W & k = \mathbb{C} \end{cases}$$

Then we can put a smooth structure on $Gr(d,V)$ as well:

Charts: Let $W \in Gr(d,V)$. Let Z be a complement of W in V (i.e. $\dim Z = n-d$ and $V = W \oplus Z$)

Then
$$\begin{array}{rccc} \Gamma_{W,Z}: & Hom(W,Z) & \longrightarrow & Gr(d,V) \\ k^{d(n-d)} \cong & \phi & \longmapsto & \text{Graph of } \phi \text{, i.e. } \{(w,\phi(w)) \in W \oplus Z = V\} \\ & 0 & \longmapsto & W. \end{array}$$

Bijective onto its image, which is
$$\{W' \mid V = W' \oplus Z\} = \{W' \mid W' \cap Z = \{0\}\}$$
a chart around W. Note different Z give different charts.

These form a smooth atlas — check that the transition fns are smooth! (See [LEE, ex 1.36])

Conclusion: $Gr(d,V)$ is a smooth mfld of dim $d(n-d)$ ($\mathbb{R}$)
$2d(n-d)$ ($\mathbb{C}$

Plücker embedding: $Gr(d,V) \xrightarrow{\iota} \mathbb{P}(\Lambda^d V)$
$W \longmapsto \mathrm{span}(w_1 \wedge \dots \wedge w_d)$ $\quad w_1 \dots w_d$ basis of W
$= \mathrm{image}(\Lambda^d W \longrightarrow \Lambda^d V)$
$\Lambda^d W \cong \mathbb{R}$. $\quad$ ind by $W \hookrightarrow V$

Image is the set of decomposable vectors in $\mathbb{P}(\Lambda^d V)$. Maybe not obvious that $w_1 \wedge \dots \wedge w_d$ determines $\mathrm{span}(w_1, \dots, w_d)$, but it can be characterized as:

Lemma. Let $\xi \in \Lambda^d V$. Then the linear map $\varphi_\xi : V \longrightarrow \Lambda^{d+1} V$, $v \mapsto v \wedge \xi$ has $\dim \ker \varphi_\xi \leq d$ with equality iff ξ is decomp as $\xi = w_1 \wedge \dots \wedge w_d$ in which case $\ker \varphi_\xi = \xi$.

Exercise. Prove this. Hint: spse $\dim \ker = d$ and use a basis of V adapted to $\ker \varphi_\xi$. p126 of [LB2018]

Cor: $Gr(d,V)$ is a projective k-alg variety (defined by the homog poly that say $\dim \ker \geq d-1$ ($\Leftrightarrow$ dim rank $\leq \dots \Leftrightarrow \det \dots = 0$)

$\Rightarrow$ COMPACTNESS!

Note: $Gr(d,n)$ or $Gr_k(d,n)$ is shorthand for $Gr(d, k^n)$.

Exercise: Find eqns for $Gr_{\mathbb{C}}(2,4)$ as a subset of $\mathbb{P}(\mathbb{C}^6)$.
$6 = \binom{4}{2} = \dim \Lambda^2 \mathbb{C}^4$ here.

Rmk. By charts constructed above, tangent spaces are

$$T_W Gr(d,V) \cong Hom(W,Z) \quad \text{where } Z \text{ is complement.}$$

This iso depends on Z however. But there is a natural iso

$$T_W Gr(d,V) \cong Hom(W, V/W). \quad (\text{of course } V/W \cong Z \ \forall \text{ complementary } Z)$$

Under this, $\psi : W \to V/W$ corresp to the velocity vector $\gamma'(0) \in T_W Gr(d,V)$ where
$\gamma(t) = \text{image}(i + t\tilde{\psi})$ where $i : V \hookrightarrow W$
$\tilde{\psi} : W \to V$ lifts $\psi : W \to V/W$.

Flags

A full flag in V is a tuple $F = (F_0, F_1, \dots, F_n)$ where

$$\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = V \text{ are subspaces, } \dim F_i = i.$$

$Flag(V) := \{\text{complete flags in } V\}$.

(Note we can omit F_0, F_n in the list of subspaces if we like)

More generally, let $(d_1, \dots, d_m)$ be positive integers w/ sum n.

A partial flag of type $(d_1, \dots, d_m)$ is a tuple $(F_0, \dots, F_m)$ where

$$\{0\} = F_0 \subset F_1 \subset \cdots \subset F_m = V$$

and $\dim F_i / F_{i-1} = d_i$. Note any tuple of nested subspaces is a flag of some type.

$\cdot \dim F_i = \dim F_{i-1} + d_i$, and d_i is the "jump" in dim.

$\mathrm{Flag}(V, d_1, \dots, d_m)$ = {partial flags of type $(d_1, \dots, d_m)$}

Full flags are of course those of type $\underbrace{(1, \dots, 1)}_{n \text{ times}}$.

$Gr(d, V) = \mathrm{Flag}(V, d)$

Thm. $\mathrm{Flag}(V, d_1, \dots, d_m)$ is a compact smooth mfld of dimension

$$\beta \cdot \sum_{1 \le i < j \le m} d_i d_j \quad \text{where} \quad \beta = \begin{cases} 1 & \text{if } k = \mathbb{R} \\ 2 & \text{if } k = \mathbb{C} \end{cases} = \underset{\substack{\uparrow \\ \text{in sense of manifolds}}}{\dim} k$$

One approach: $\mathrm{Flag}(V, d_1, \dots, d_m) \hookrightarrow Gr(d_1, n) \times Gr(d_1 + d_2, n) \times \cdots \times Gr(d_1 + \cdots + d_{m-1}, n)$

study image.

We'll take a different approach: understanding $\mathrm{Flag}(\cdots)$ as a homogeneous space of a Lie group.

Obs. $\mathrm{Aut}(V)$ acts transitively on $\mathrm{Flag}(V)$ or $\mathrm{Flag}(V, d_1, \dots, d_m)$